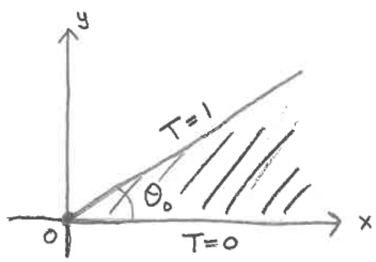
SNAP 2017. Laplace's equation and conformal maps.

Problem Set 2

1. Let $H(u,v)$ be a harmonic function (i.e. $H_{uu} + H_{vv} = 0$). Let $f(z) = u(x,y)+iv(x,y)$ be a holomorphic function, where we are writing $z = x+iy$ as usual. Show directly using the Chain rule and the Cauchy-Riemann equations that $h(x,y) = H(u(x,y), v(x,y))$ is a harmonic function of x and y where defined.

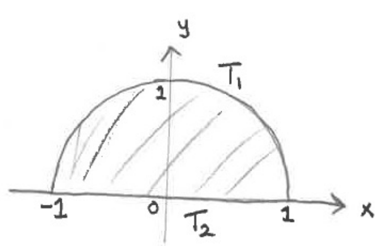
2. Fix an angle θ_0 with $0 < \theta_0 < \pi/2$ and consider the wedge shaped region as shown. Find the steady state heat distribution $T(x,y)$ if the top edge is held at temperature $T = 1$ and the bottom edge is held at temperature $T = 0$.



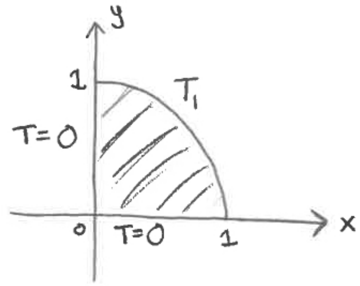
3. Find the steady state heat distribution $T(x,y)$ on a half disk $\{x^2 + y^2 < 1,\ y > 0\}$ if the top semi-circular edge is held at a temperature T_1 and the bottom edge $\{(x,0) \mid -1 < x < 1\}$ is held at a temperature T_2.

   Show that the isotherms (curves with $T =$ constant) are arcs of circles with center on the y-axis, and which pass through the points $(-1,0)$ and $(1,0)$.

   Hint: consider the full disk with the lower semi-circle held at temperature T_3 *with* $T_2 = \frac{1}{2}(T_1 + T_3)$.

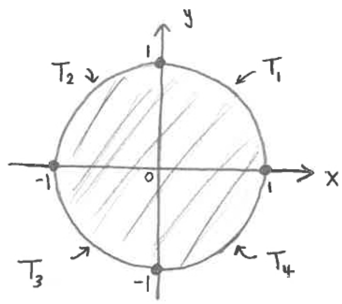


4. Find the steady state heat distribution $T(x, y)$ on the quarter disk $\{x^2 + y^2 < 1,\ y > 0,\ x > 0\}$ if the quarter-circular edge is held at a temperature T_1 and the two straight edges along the x and y-axes are held at a temperature zero.

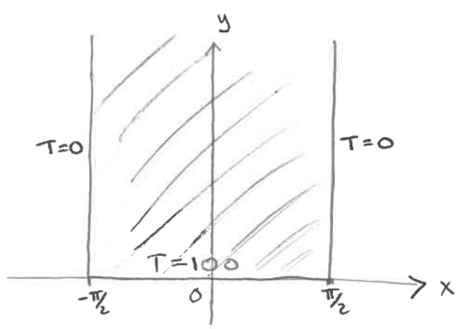


5. Find the steady state heat distribution $T(x, y)$ on the unit disk $\{x^2+y^2 < 1\}$ if the quarter circles are held at constant temperatures T_1, T_2, T_3, T_4 as shown.

Hint: map to the upper half plane and consider solutions of the form $w \mapsto Arg(w - a)$.



6. Find the steady state heat distibution $T(x, y)$ on the semi-infinite vertical slab $\{(x, y) \mid -\pi/2 < x < \pi/2,\ y > 0\}$, with the bottom edge held at temperature $T = 100$ and the sides held at temperature zero.



7. Let $V(x, y)$ be the potential function for the electric field for a conducting laminar plate corresponding to the lunar domain

$$\{x^2 + y^2 < 1\} \cap \{(x - 1/2)^2 + y^2 > 1/4\}$$

with boundary values $V = 0$ on the unit circle and $V = 1$ on the smaller circle $(x - 1/2)^2 + y^2 = 1/4$. Solve for $V(x, y)$ and show that the equipotential curve $V(x, y) = c$ is a circle centered at $(c/(1 + c), 0)$ with radius $1/(1 + c)$.

Hint: map the domain to a horizontal strip with a Möbius transformation that sends 1 *to* ∞